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A POSTERIORI ERROR ESTIMATION FOR HIERARCHIC MODELS OF ELLIPTIC BOUNDARY VALUE PROBLEMS ON THIN DOMAINS

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A posteriori error estimation for hierarchic models of elliptic boundary value problems on thin domains

by

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Abstract

The boundary value problem of heat conduction in a three dimensional, laminated plate is approximated by a hierarchy of two dimensional models. Computable a-posteriori indicators and estimators of the modelling error in various norms are derived and their local spectral and asymptotic exactness is proved. Sharp estimates for their effectivity indices are also obtained.

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INTRODUCTION

The modelling of the elastic behavior of thin objects has a long history. The main idea is to replace the particular three dimensional problem by a two or one dimensional one, which is easier to solve. Such approaches were already proposed in the first half of the nineteenth century by S. Germain [1] and G. Kirchhoff [2]. Since then many approaches were proposed. For some surveys we refer for example to [3] and [4]. The derivation of these models is based on physical considerations, a mathematical analysis of various degrees of rigor or on the asymptotic analysis of the three dimensional problem as the thickness of the structure tends to zero. We refer to [5] and references there for this approach. In general all the methodologies can be understood as the application of a dimensional reduction approach.

This approach leads to an approximate solution of the original higher dimensional problem. Hence an error estimate is needed. There are presently various *a priori* error estimates (see e.g. [6], [7]) or estimates of asymptotic character (see e.g. [5]) available.

Nevertheless in today's computational environments we need

a) an accurate and computable *a posteriori* estimate of the difference (error) between the exact solution of the original three dimensional problem and the dimensionally reduced one, the *modelling error*, and

b) a procedure which leads to the construction of a hierarchy of dimensionally reduced models which allow to solve the original three dimensional problem with a prescribed given tolerance or accuracy and this procedure has to be adaptive (we remark that in contrast to the classical approaches, the adaptive approach leads to models which are not uniform through the entire domain).

As in today's adaptive finite element approaches, the fundamental part of the adaptive procedure are a posteriori error estimates based on *local indicators* which should be of high quality. By this we mean that the estimator and the indicator have to be robust, i.e. their effectivity index should be reasonably well bounded from below and above for a large class of solutions and should be asymptotically exact for more restrictive classes of solutions.

The present paper addresses these questions for the heat conduction problem on a thin domain when the material is homogeneous or laminated. It gives a computable a-posteriori estimate for the modelling error measured in the (weighted) energy and L_2-norm. The indicators are local and hence very well suited for adaptive approaches. Upper and lower bounds of their effectivity indices are also obtained. The adaptive procedures based on this approach will be discussed elsewhere (see e.g. [8]). Let us now outline the contents of this paper.

In Section 1 we introduce the formulation of the problem and the main notations. In Section 2 we introduce the hierarchic models and some of their basic properties. Section 3 addresses some abstract functional analytical results which will be employed later. Section 4 introduces the a-posteriori estimator and proves its basic properties, especially the upper and lower bound for its effectivity index. Section 5 analyses the asymptotic exactness of the estimator as the thickness of the plate $d \to 0$ and Section 6 analyses the spectral asymptotic exactness when the degree of the model increases. Sections 4, 5 and 6 address the estimator for the modelling error measured in a weighted energy norm where the weight is exponential. Section 7 addresses the error estimate for the L_2 measure of the modelling error. Section 8 generalizes the estimator to laminated materials and the final Section 9 presents a simple numerical example to illustrate the sharpness of our estimates.

1. Notation and Problem Formulation.

By $\omega \subset \mathbb{R}^2$ we denote a bounded domain with a piecewise smooth boundary γ satisfying the cone condition. With a positive thickness parameter d and ω we associate the three dimensional domain

$$\Omega = \omega \times (-d/2,\ d/2)$$

with lateral boundary

$$\Gamma = \gamma \times (-d/2,\ d/2)$$

and the faces

$$R_\pm = \{(x_1,x_2,x_3) \mid (x_1,x_2) \in \omega\ ,\ x_3 = \pm d/2\}.$$

In Ω we consider a heat conduction problem with prescribed heat fluxes $f^\pm$ on the faces, i.e.

$$(1.1)\qquad \begin{aligned} Lu &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ D_n u &= f^\pm && \text{on } R_\pm. \end{aligned}$$

where the operator L is (in the sense of distributions) given by

$$(1.2)\qquad -Lu = \frac{\partial}{\partial x_3}\left[a\left(\frac{2x_3}{d}\right)\frac{\partial u}{\partial x_3}\right] + b\left(\frac{2x_3}{d}\right)\nabla_x \cdot \left[C(x)\nabla_x u\right],$$

where $\nabla_x = \left[\frac{\partial}{\partial x_1}\ ,\ \frac{\partial}{\partial x_2}\right]^T$, $x = (x_1,x_2)$; $a(\cdot)$, $b(\cdot) \in L^\infty(-1,1)$ are even functions independent of d and satisfy

$$(1.3)\qquad 0 < \underline{A} \le a(z)\ ,\ 0 < \underline{B} \le b(z)\ .$$

The matrix-function $C(x)$ is symmetric and uniformly positive definite, i.e. there exist constants $0 < \underline{C} \le \overline{C} < \infty$ so that

$$(1.4)\qquad \underline{C}|\xi|^2 \le \xi^T C(x)\xi \le \overline{C}|\xi|^2$$

for all $\xi \in \mathbb{R}^2$, $x \in \bar{\omega}$; $C(x)$ has C^∞ coefficients and the boundary operator $D_n u$ is the (distributional) exterior conormal derivative on $R_{\pm}$.

To cast (1.1) into the weak form we introduce the Sobolev space

$$H := \left\{u \in H^1(\Omega) \,\middle|\, \text{trace } u = 0 \text{ on } \Gamma\right\} \tag{1.5}$$

and define the bilinear form $B(\cdot,\cdot) : H \times H \to \mathbb{R}$ and the functional $F(\cdot)$: $H \to \mathbb{R}$ by

$$B(u,v) = \int_\Omega \left\{a\left(\frac{2x_3}{d}\right) \frac{\partial u}{\partial x_3} \frac{\partial v}{\partial x_3} + b\left(\frac{2x_3}{d}\right) (\nabla_x u)^T C(x) \nabla_x v\right\} dx_1 dx_2 dx_3 \tag{1.6}$$

and

$$F(v) = \int_\omega \left\{f^+(x_1,x_2)\, v\, (x_1,x_2,\tfrac{d}{2}) + f^-(x_1,x_2)\, v\, (x_1,x_2,-\tfrac{d}{2})\right\} dx_1 dx_2, \tag{1.7}$$

respectively.

Then the weak form of (1.1) reads: Find $u \in H$ such that

$$B(u,v) = F(v) \qquad \forall v \in H. \tag{1.8}$$

Under the assumptions (1.3), (1.4) there exists a unique weak solution of (1.8) provided that

$$f^+,\ f^- \in L^2(\omega) \tag{1.9}$$

(this assumption could be weakened, but is sufficient for our purpose).

2. Hierarchical Modelling.

We will approximate the boundary value problem (1.5)-(1.9) by a sequence of two dimensional problems on ω, the *hierarchy of plate models*, which we now define.

Denote by

$$\mathcal{P} = \{\omega_i \,|\, \omega_i \subseteq \omega\ ,\ 1 \le i \le n\} \tag{2.1}$$

a collection of n domains with piecewise smooth boundaries $\partial\omega_i$ such that

$\omega_i \cap \omega_j = \phi$ if $i \neq j$ and $\bar{\omega} = \bigcup_{i=1}^{n} \bar{\omega}_i$ ($\mathcal{P}$ could be, for example, a triangulation of ω). For a vector q of nonnegative integers

$$q = \{q_1,\ldots,q_n\}\ ,\ q_i \geq 0 \tag{2.2}$$

and a dense sequence

$$\left\{\psi_j(z)\right\}_{j=0}^{\infty} \subset H^1(-1,1) \tag{2.3}$$

of linearly independent functions we define

$$S(\mathcal{P},q) := \left\{u \in H \,\middle|\, u\big|_{\omega_i} = \sum_{j=0}^{q_i} U_j^{(i)}(x_1,x_2)\ \psi_j\left[\frac{2x_3}{d}\right],\ \omega_i \in \mathcal{P}\right\}. \tag{2.4}$$

Then $S(\mathcal{P},q) \subset H$ and the $(\mathcal{P},q)$-plate model is the boundary value problem: Find $u(\mathcal{P},q) \in S(\mathcal{P},q)$ such that

$$B(u(\mathcal{P},q),v) = F(v) \qquad \forall v \in S(\mathcal{P},q), \tag{2.5}$$

i.e. $u(\mathcal{P},q)$ is the (energy) projection of the weak solution u onto $S(\mathcal{P},q)$. Hence (2.5) constitutes an elliptic boundary value problem on ω for the coefficient functions $U_j^{(i)}(x_1,x_2)$ in (2.4).

The selection of the functions ψ_j in (2.3) completely determines the $(\mathcal{P},q)$-model and has been investigated in [9], from where we quote the following results. Define $\psi_{2j}(z) = \psi_{2j}(-z)$, $j = 0,1,\ldots$ recursively

$$\int_{-1}^{1} a(z)\psi_0'(z)v'(z)dz = 0\ , \tag{2.6}$$

$$\int_{-1}^{1} a(z)\psi_{2j}'(z)v'(z)dz + \int_{-1}^{1} b(z)\psi_{2j-2}(z)v(z)dz = \delta_j(v) \tag{2.7}$$

for all $v \in H^1(-1,1)$, $j \in \mathbb{N}$, where

$$\delta_j(v) = \begin{cases} v(1) + v(-1) & \text{if} \quad j = 1 \\ 0 & \text{else} \end{cases}$$

and $\psi_{2j+1}(z) = -\psi_{2j+1}(-z)$, $j = 0,1,2,\dots$ by

$$\text{(2.8)} \qquad \int_{-1}^{1} a(z)\psi'_{2j+1}(z)v'(z)dz + \int_{-1}^{1} b(z)\psi_{2j-1}(z)v(z)dz = \tilde{\delta}_j(v)$$

for all $v \in H^1(-1,1)$, $j \in \mathbb{N}$, where

$$\tilde{\delta}_j(v) = \begin{cases} v(1) - v(-1) & \text{if} \quad j = 1 \\ 0 & \text{else} \end{cases}$$

and

$$\text{(2.9)} \qquad \psi_{-1} = 0.$$

Remark 2.1. It is not hard to see that (2.6)-(2.9) determines the sequence uniquely (the nonuniqueness in the solutions of (2.7), (2.8) is taken care of by requiring the compatibility condition in the subsequent step). Moreover, it was shown in [9] that

$$\text{(2.10)} \qquad \{\psi_j(z)\}_{j=0}^{\infty} \text{ is dense in } H^1(-1,1). \qquad \square$$

Remark 2.2. If $a(z)$ and $b(z)$ are constant, $\psi_j(z)$ is a polynomial. Table 1 lists the first four ψ_j.

j	$\psi_j(z)$
0	1
1	z
2	$(3z^2 - 1)/6$
3	$(z^3 - 3z)/6$
4	$(15z^4 - 30z^2 + 7)/360$

Table 1. The first $\psi_j(z)$ for $a = b = 1$, $0 \le j \le 4$.

If $a(z)$, $b(z)$ are piecewise constant, $\psi_j(z)$ is a piecewise polynomial. This is the situation for sandwich materials.

With the choice (2.6)-(2.9) of $\psi_j(z)$, the *modelling error*

$$e(\mathcal{P},q) := u - u(\mathcal{P},q) \tag{2.11}$$

is, for $\mathcal{P} = \{\omega\}$ and $q = N$, of optimal asymptotic order as $d \to 0$, provided the data f^+, f^- are sufficiently regular in ω and satisfy certain compatibility conditions on the edges $\partial\Gamma = \gamma \times \{\pm d\}$ which ensure the absence of boundary layers (see [10], for example). We emphasize at this point that due to (2.10) the error $e(\mathcal{P},q)$ will also tend to zero for fixed $d > 0$, if $\min\{q_i\} \to \infty$ in contrast to the error in models obtained by asymptotic analysis.

In the following sections we will derive computable a posteriori error estimators for the modelling error measured in the energy norm

$$\|e(\mathcal{P},q)\|_{E(\Omega)} = (B(e,e))^{1/2} \tag{2.12}$$

in terms of the residual data on the faces $R_\pm$. The following property of $e(\mathcal{P},q)$ will prove to be important.

Theorem 2.1. For every $(\mathcal{P},q)$ we have

$$\int_{-d/2}^{d/2} b\left(\frac{2x_3}{d}\right) e(x_1,x_2,x_3)\,dx_3 = 0 \quad \text{a.e.} \quad (x_1,x_2) \in \omega. \tag{2.13}$$

Proof. It follows from (2.6) that $\psi_0(z) = \text{const}$. Due to (2.5) we have

$$B(e(\mathcal{P},q),\, v) = 0 \quad \forall v \in S(\mathcal{P},q)$$

and, since $X(x)\psi_0\left(\frac{2x_3}{d}\right) \in S(\mathcal{P},q)$ for all $(\mathcal{P},q)$, we find with Fubini's theorem

$$(2.14) \qquad 0 = \int_\omega \nabla_x X(x_1,x_2)\cdot C(x_1,x_2)\ \nabla_x \int_{-d/2}^{d/2} b\left[\frac{2x_3}{d}\right] e(x_1,x_2,x_3)\,dx_3\,dx_1 dx_2$$

for all $X \in \mathring{H}^1(\omega) = \{u \in H^1(\omega) \mid \text{trace } u \text{ on } \gamma = 0\}$.

Let

$$\psi(x_1,x_2) = \int_{-d/2}^{d/2} b\left[\frac{2x_3}{d}\right] e(x_1,x_2,x_3)\,dx_3 .$$

Since $e \in H$, $\psi \in \mathring{H}^1(\Omega)$. Using $X = \psi$ in (2.14) we conclude that $\psi = 0$ which was to be proven. □

We shall derive in Section 4 computable guaranteed upper estimates for the modelling error (2.12) which are asymptotically exact. Moreover, our estimators also give information about the local contributions from ω_i to $\|e\|_{E(\Omega)}$. As a tool we shall use certain exponentially weighted spaces, which we analyze next.

Remark 2.3. Here and in what follows we assume that the elliptic system (2.5) of differential equations for the unknown coefficient functions $U_j^{(i)}$ is solved exactly. However, usually only an approximate solution can be obtained. Our results remain nevertheless valid, if the approximate solution has a sufficiently high accuracy. In computational practice one usually works with a finite element approximation of $U_j^{(i)}$. In this case the a posteriori estimation of modelling error can be used to determine the desirable accuracy of the finite element approximations of $U_j^{(i)}$. □

3. Some abstract results.

Let H_1, H_2 be two reflexive Banach spaces furnished with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Further, let $B(u,v)$ be a bilinear form defined on $H_1 \times H_2$. We will call the bilinear form (C,γ)-*regular* if there exist constants $0 < C, \gamma < \infty$ so that

$$|B(u,v)| \le C\|u\|_1 \|v\|_2, \tag{3.1}$$

$$\inf_{\|u\|_1=1} \sup_{\|v\|_2=1} |B(u,v)| \ge \gamma, \tag{3.2}$$

$$\text{for any } v \ne 0,\ v \in H_2, \quad \sup_{\|u\|_1=1} |B(u,v)| > 0 . \tag{3.3}$$

Bilinear forms satisfying (3.1) - (3.3) have the following properties.

a) Let $f \in (H_2)'$ (i.e. f is a bounded, linear functional on H_2), then there exists exactly one $u \in H_1$ such that

$$B(u,v) = f(v),\ \forall v \in H_2.$$

b) If

$$\sup_{\|v\|_2=1} |B(u,v)| \le A, \tag{3.4}$$

then

$$\|u\|_1 \le \frac{A}{\gamma} .$$

Let us consider now some special cases which will be important later.

Let $0 < \varphi(x_1,x_2) \in W^{1,\infty}(\omega)$ denote a (strictly positive) weight function in ω. We define

$$H_\varphi = \{e \in H \mid e \text{ satisfies } (2.13)\}$$

and furnish H_φ with the weighted energy norm defined by

$$\|e\|_\varphi^2 = \int_\Omega \varphi^2(x_1,x_2) \left\{a\left(\frac{2x_3}{d}\right)\left(\frac{\partial e}{\partial x_3}\right)^2 + b\left(\frac{2x_3}{d}\right) \nabla_x e^T C \nabla_x e\right\} dx_1 dx_2 dx_3 . \tag{3.6}$$

The following Lemma will be used repeatedly.

<u>Lemma 3.1</u>. Assume that $u \in H_\varphi$. Then

$$(3.7) \qquad \int_{\sigma\times(-d/2,d/2)} \varphi^2 b\left(\frac{2x_3}{d}\right)u^2dx_1dx_2dx_3 \le \Lambda^2d^2 \int_{\sigma\times(-d/2,d/2)} \varphi^2 a\left(\frac{2x_3}{d}\right)\left(\frac{\partial u}{\partial x_3}\right)^2 dx_1dx_2dx_3$$

for all open subsets $\sigma \subseteq \omega$. Here Λ is given by

$$\frac{1}{\Lambda^2} = \inf_{\psi\in H^1(-1,1)} \frac{\int_{-1}^{1} a(z)(\psi')^2dz}{\int_{-1}^{1} b(z)\psi^2dz}$$

and the infimum is taken over all

$$\psi \in H^1(-1,1) \cap \left\{\psi \middle| \int_{-1}^{1} b(z)\psi(z)dz = 0\right\}.$$

Proof. Assume that $u \in C^\infty(\bar{\Omega}) \cap H_\varphi$. Then we have for all $x \in \omega$ the bound

$$(3.8) \qquad \int_{-d/2}^{d/2} b\left(\frac{2x_3}{d}\right)|u|^2dx_3 \le \Lambda^2d^2 \int_{-d/2}^{d/2} a\left(\frac{2x_3}{d}\right) \left(\frac{\partial u}{\partial x_3}\right)^2 dx_3$$

by the definition of Λ and a scaling argument. Multiplying both sides of (3.8) by φ^2 and integrating over σ we get (3.7) for u and a density argument completes the proof. $\square$

<u>Remark 3.1</u>. For $a = b = 1$ we find $\Lambda = \frac{2}{\pi}$. If ψ is symmetric in z then $\Lambda = \frac{1}{\pi}$.

<u>Theorem 3.1</u>. Let $0 < \varphi(x_1,x_2) \in W^{1,\infty}(\omega)$ and assume that

$$Q := \max_{i=1,2} \left\| \frac{\partial\varphi^2}{\partial x_i}\bigg/\varphi^2\right\|_{L^\infty(\omega)} < \infty.$$

Define $H_1 = H_\varphi$, $H_2 = H_{1/\varphi}$. Then the bilinear form (1.6) is $(1,\gamma)$ regular on $H_1 \times H_2$ with

$$(3.9)\quad \gamma \ge \gamma_0 \;:= \left[1 - d\Lambda Q\sqrt{\bar{C}/2}\right]\left[1 + d\sqrt{2\bar{C}}\,\Lambda Q\left[1 + d\sqrt{2\bar{C}}\,\Lambda Q\right]\right]^{-1/2}.$$

Proof:

1) $|B(u,v)| \le \|u\|_\varphi \|v\|_{1/\varphi}$ follows immediately from Schwarz' inequality.

2) Let us show (3.2). For $u \in H_\varphi$ define $v = \varphi^2 u$. Then $v \in H_{1/\varphi}$ and

$$\nabla_x v = \varphi^2 \nabla_x u + u\nabla_x(\varphi^2), \quad \frac{\partial v}{\partial x_3} = \varphi^2 \frac{\partial u}{\partial x_3}.$$

Hence, denoting the volume element by dx, we find

$$(3.10)\qquad B(u,u) = \|u\|_\varphi^2 + \int_\Omega \left\{b\left(\frac{2x_3}{d}\right) u\,\nabla_x(\varphi^2)^T C(x)\nabla_x u\right\} dx,$$

and we estimate for every $\varepsilon > 0$

$$\left|\int_\Omega \left\{b\left(\frac{2x_3}{d}\right) u\,\nabla_x(\varphi^2)^T C(x)\nabla_x u\right\} dx\right|$$

$$\le Q\sqrt{\bar{C}/2}\left\{\frac{1}{\varepsilon}\int_\Omega b\left(\frac{2x_3}{d}\right) u^2\varphi^2 dx + \varepsilon\int_\Omega \varphi^2 b\left(\frac{2x_3}{d}\right)\nabla_x u^T C(x)\nabla_x u\, dx\right\}.$$

Utilizing Lemma 3.1 with $\sigma = \omega$, we arrive at

$$\int_\Omega \left\{b\left(\frac{2x_3}{d}\right) u\,\nabla_x(\varphi^2)^T C(x)\nabla_x u\right\} dx$$

$$\le Q\sqrt{\bar{C}/2}\left\{\frac{\Lambda^2 d^2}{\varepsilon}\int_\Omega \varphi^2 a\left(\frac{2x_3}{d}\right)\left(\frac{\partial u}{\partial x_3}\right)^2 dx + \right.$$

$$\left.\varepsilon\int_\Omega \varphi^2 b\left(\frac{2x_3}{d}\right)\nabla_x u^T C(x)\nabla_x u\, dx\right\}.$$

Selecting $\varepsilon = \Lambda d$ yields

$$B(u,v) \ge \left(1 - d\Lambda Q\sqrt{\bar{C}/2}\right)\|u\|_\varphi^2.$$

Further,

$$\|v\|^2_{1/\varphi} = \|u\|^2_\varphi + \int_\Omega \varphi^{-2} b\left(\frac{2x_3}{d}\right)\left\{2\varphi^2 u\nabla_x(\varphi^2)^T C(x)\nabla_x u + u^2\nabla_x(\varphi^2)^T C(x)\nabla_x(\varphi^2)\right\} dx$$

$$\le \|u\|^2_\varphi + d\,\Lambda Q\sqrt{\overline{C}}\ \|u\|^2_\varphi + 2d^2\Lambda^2\overline{C}\int_\Omega \varphi^2 a\left(\frac{2x_3}{d}\right)\left(\frac{\partial u}{\partial x_3}\right)^2 dx$$

$$\le \left[1 + d\sqrt{2\overline{C}}\ \Lambda Q\left[1 + d\sqrt{2\overline{C}}\ \Lambda Q\right]\right] \|u\|^2_\varphi$$

from where get (3.9). □

Remark 3.2. We observe that

$$Q = 2 \max_{i=1,2} \left\| \frac{\partial\varphi}{\partial x_i} \Big/ \varphi \right\|_{L^\infty(\omega)} ,$$

and, from (3.9), we can select φ in particular so that $\alpha := Q\Lambda\sqrt{\overline{C}/2}\, d < 1$ and get

(3.11) $$\gamma \ge (1 - \alpha)\ (1 + 2\alpha(1 + 2\alpha))^{-1/2}$$

4. A posteriori estimation of the modelling error.

In this section we assume that the $(\mathcal{P},q)$-model (2.5) has uniform order q and that its exact solution $u(\mathcal{P},q)$ is known. We will be interested in computable estimators for (2.12), the modelling error in energy norm. To avoid obscuring the main ideas behind technicalities, we assume throughout this section

(4.1) $$a(z) = b(z) = 1\ ,\ C(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

i.e. L in (1.1) is the Laplace operator and $D_n = \partial/\partial n$ is the outside normal derivative. All results apply with minor modifications in the proofs which we will present in Section 8 also to (1.1). It is convenient to write $u = u_1 + u_2$ where

(4.2) $$u_1(x_1,x_2,x_3) = -u_1(x_1,x_2,-x_3),\ u_2(x_1,x_2,x_3) = u_2(x_1,x_2,-x_3).$$

The u_i satisfy : $u_i \in H_i$ such that

(4.3) $$B(u_i,v) = F_i(v) \qquad \forall v \in H_i,\quad i = 1,2$$

where

$$F_1(v) = \int_\omega f_1(x_1,x_2)(v(x_1,x_2,d/2) - v(x_1,x_2,-d/2))\,dx_1dx_2\ ,$$

$$F_2(v) = \int_\omega f_2(x_1,x_2)(v(x_1,x_2,d/2) + v(x_1,x_2,-d/2))\,dx_1dx_2\ ,$$

(4.4) $$f_1(x_1,x_2) = \frac{1}{2}(f^+ - f^-)(x_1,x_2), f_2(x_1,x_2) = \frac{1}{2}(f^+ + f^-)(x_1,x_2)\ ,$$

$H_i = \{u \in H \mid u$ is antisymmetric (symmetric) in x_3 for $i = 1(i=2)\}$.
Obviously, the spaces H_1 and H_2 are orthogonal in energy, i.e.

(4.4) $$B(u,v) = 0 \qquad \forall\ u \in H_1,\ \forall v \in H_2\ ,$$

and $u(\mathcal{P},q) = u_1(\mathcal{P},q) + u_2(\mathcal{P},q)$, each of which can be obtained by energy projection of u_i onto

(4.5) $$S_i(\mathcal{P},q) := S(\mathcal{P},q) \cap H_i,\ i = 1,2.$$

Further, from (4.4) we get also

(4.6) $$\|e(\mathcal{P},q)\|^2_{E(\Omega)} = \|e_1(\mathcal{P},q)\|^2_{E(\Omega)} + \|e_2(\mathcal{P},q)\|^2_{E(\Omega)}\ ,$$

where $e_i(\mathcal{P},q) = u_i - u_i(\mathcal{P},q)$, $i = 1,2$.
Since $S(\mathcal{P},q)$ in (2.4) depends only on $\mathrm{span}\{\psi_j(z)\}_{j=0}^q = \Pi_q(-1,1)$ (see

Remark 2.2), we will assume below for convenience that

$$\psi_j(z) = L_j(z) \;, \tag{4.7}$$

where L_j denotes the jth Legendre polynomial on $(-1,1)$.

All our *a-posteriori estimators* $\mathcal{E}$ for (4.6) are of the form

$$\mathcal{E}(u_i(\mathcal{P},q)) = \left[\iint_\omega |\eta_i(x_1,x_2)|^2 dx_1 dx_2\right]^{\frac{1}{2}} \;, \quad i = 1,2. \tag{4.8}$$

Here $\eta_i(x_1,x_2)$ is called *indicator function.*

Let $\|\circ\|$ be any norm on H and $\mathcal{E}$ in (4.8) an a-posteriori error estimator for $\|e(\mathcal{P},q)\|$. Then we define the effectivity index Θ corresponding to $\mathcal{E}$ and $\|\circ\|$ by

$$\Theta := \frac{\mathcal{E}(u(\mathcal{P},q))}{\|e(\mathcal{P},q)\|} \;. \tag{4.9}$$

We say that $\mathcal{E}$ is a *guaranteed upper estimator*, if $\Theta \ge 1$ for all u. The estimator $\mathcal{E}$ is (κ_1,κ_2)*-proper* with respect to a class T of data, if

$$0 < \kappa_1 \le \Theta \le \kappa_2 < \infty \qquad \forall f \in T.$$

Further, $\mathcal{E}$ is *asymptotically exact on* T if

$$\Theta \longrightarrow 1 \quad \text{as} \quad d \longrightarrow 0^+ \qquad \forall f \in T, \tag{4.10$'$}$$

and $\mathcal{E}$ is *spectrally exact on* T if

$$\Theta \longrightarrow 1 \quad \text{as} \quad q \longrightarrow \infty \qquad \forall f \in T. \tag{4.10$''$}$$

Finally, $\mathcal{E}$ is locally asymptotically (resp. spectrally) exact on T, if (4.10)′ (resp. (4.10)″) hold with the norm $\|\circ\|_\varphi$ defined in (3.6) where the weight function $\varphi(x_1,x_2)$ is given by

$$\varphi(x_1,x_2) := \exp \frac{\alpha}{d^\rho}\left\{|x_1-x_2^\circ| + |x_2-x_2^\circ|\right\}, \; 0 < \rho \le 1, \; 0 \le \alpha < 1/(\Lambda\sqrt{2C}\,)$$

and $(x_1^\circ,x_2^\circ) \in \omega$ is arbitrary.

We begin the analysis of the estimator $\mathcal{E}$ for the case $\mathcal{P} = \{\omega\}$ and the energy norm and consider the L^2-norm later in Section 7. Whenever the order of the model is uniform throughout ω, we shall omit the index $\mathcal{P}$. Due to (4.6) we can derive the indicator functions η_i, $i = 1,2$, separately. We start by observing that, due to Theorem 2.1, the errors $e_i(q) \in H_\varphi$, $i = 1,2$, defined in (4.6) satisfy

(4.12) $$B(e_i(q),v) = R_i(v) \qquad \forall v \in H$$

and

(4.13) $$B(e_i,v) = 0 \qquad \forall v \in S_i(q)$$

where

$$R_i(v) = \int_\omega r_i(x_1,x_2)\ (v(x_1,x_2,d/2) \pm v(x_1,x_2,-d/2))dx_1dx_2$$

$$+ \int_\Omega v(x_1,x_2,x_3)\ \Delta u_i(q)dx_1dx_2dx_3, \quad i = 1,2$$

and $-,+$ correspond to $i = 1,2$, respectively. Here

(4.14) $$r_i(x_1,x_2) = f_i(x_1,x_2) - \frac{\partial u_i(q)}{\partial n}(x_1,x_2,d/2), \quad i = 1,2.$$

<u>Remark 4.1</u>. For $f_i \in H^s(\omega)$, $s \geq 0$, we have

(4.15) $$r_i \in H^{\min(s,1+\varepsilon)}(\omega)$$

where $\varepsilon > 0$ is determined by the maximal regularity of the solution to the Dirichlet problem for Δ_x in ω (e.g. if ω is a slit domain, $0 \leq \varepsilon < 1/2$). The unknown coefficient functions in (2.4) satisfy the elliptic system

$$-\frac{d}{2}\,A\,\Delta_x U + \frac{2}{d}\,B\,U = c^+f^+ - c^-f^- \quad \text{in } \omega$$

$$U = 0 \qquad \text{on } \partial\omega.$$

The matrices A and B are independent of d and given by

$$A_{ij} = \int_{-1}^{1} \psi_i\psi_j dz, \quad B_{ij} = \int_{-1}^{1} \psi_i'\psi_j' dz$$

and

$$c^{\pm} = \{\psi_0(\pm 1), \ldots, \psi_q(\pm 1)\}^T$$ □

We calculate next a simplified expression for $R_i(v)$ which we will use repeatedly below.

Lemma 4.1. Let $i = 1$, $q = 2m+1$ or $i = 2$, $q = 2m$ then

$$\text{(4.16)} \quad \begin{aligned} R_i(v) = \int_\omega r_i(x_1,x_2) \Big\{ & v(x_1,x_2,d/2) \pm v(x_1,x_2,-d/2) \\ & - \int_{-d/2}^{d/2} v(x_1,x_2,x_3) \frac{d}{dx_3}\left[L_{q+1}\left(\frac{2x_3}{d}\right)\right] dx_3 \Big\} dx_1 dx_2. \end{aligned}$$

Proof: Let $i = 1$, $q = 2m+1$. Then

$$\text{(4.17)} \quad \Delta u_1(q) = \sum_{j=0}^{2m+1} A_{1j}(x_1,x_2)\, L_j\left(\frac{2x_3}{d}\right)$$

for some $A_{1j} \in H^{-1}(\omega)$. To determine A_{1j}, we use

$$R_1(v) = 0 \qquad \forall v \in S_1(q) \cup H_2.$$

We select $v = V(x_1,x_2)\, L_{2k}\left(\frac{2x_3}{d}\right) \in H_2$ with arbitrary $V \in \mathring{H}^1(\omega)$ and get $A_{1j} = 0$ for even j. For $j = 2k+1$, $0 \le k \le m$, we find

$$0 = \int_\omega V(x_1,x_2)\left\{2r_1 + A_{1,2k+1}\frac{d}{2}\int_{-1}^{1}\left[L_{2k+1}(z)\right]^2 dz\right\} dx_1 dx_2.$$

Since we get $V \in \mathring{H}^1(\omega)$ is arbitrary, we get

$$A_{1,2k+1} = -\frac{2}{d}(2(2k+1)+1)\, r_1$$

(note that due to Remark 4.1, $A_{1,2k+1} \in L^2(\omega)$).

Hence

$$\Delta u_1(q) = -\frac{2}{d}\, r_1(x_1,x_2) \sum_{k=0}^{m} (4k+3)\, L_{2k+1}\left(\frac{2x_3}{d}\right)$$
$$= -\, r_1(x_1,x_2)\, \frac{d}{dx_3}\left[L_{2m+2}\left(\frac{2x_3}{d}\right)\right].$$

For $i = 2$ and $q = 2m$, one proceeds analogously. □

We derive next the estimator $\mathcal{E}$. We start with the observation that from Theorems 2.1, 3.1 we have the bound

$$(4.18) \qquad \gamma_0\, \|u_i - u_i(q)\|_{\varphi} \le \sup_{0 \ne \|v\|_{1/\varphi}} \frac{|B(u_i - u_i(q), v)|}{\|v\|_{1/\varphi}}, \quad i = 1,2$$

where γ_0 is as in (3.9). With (4.12) thus

$$\gamma_0^2 \|e_1(q)\|_{\varphi}^2 \le \sup_{0 \ne \|v\|_{1/\varphi}} \frac{(R_1(v))^2}{\|v\|_{1/\varphi}^2}$$
$$\le \sup_{0 \ne \|v\|_{1/\varphi}} \frac{(R_1(v))^2}{\int_{\Omega} \varphi^{-2} \left(\frac{\partial v}{\partial x_3}\right)^2 (x_1,x_2,x_3)\,dx}$$
$$\le \sup_{0 \ne \|v\|_{1/\varphi}} \left[\int_{\omega} \varphi r_1\, \Phi_1[v]\, dx_1 dx_2\right]^2$$

where

$\left[\Phi_1[v]\right](x_1,x_2) :=$

$$(4.19) \qquad \frac{v(x_1,x_2,d/2) \mp v(x_1,x_2,-d/2) - \int_{-d/2}^{d/2} v(x_1,x_2,x_3)\, \frac{d}{dx_3}\left[L_{q+1}\left(\frac{2x_3}{d}\right)\right] dx_3}{\left[\int_{-d/2}^{d/2} \left(\frac{\partial v}{\partial x_3}\right) dx_3\right]^{1/2}}$$

Hence we obtain, using Jensen's inequality, that

$$\gamma_0^2 \|e_1(q)\|_\varphi^2 \le \int_\omega \varphi^2 r_1^2 \sup_{0 \ne v \in M} (\Phi_1[v])^2 dx_1 dx_2 \tag{4.20}$$

where the supremum is taken over

$$M := L^2(\omega, H^1(-d/2, d/2)) \cap \left\{ v \,\Big|\, \int_{-d/2}^{d/2} v \, dx_3 = 0 \quad \text{a.e.} \quad (x_1, x_2) \in \omega \right\} \tag{4.21}$$

(see [11] for the definition of anisotropic Sobolev spaces).

Since Φ_i is strictly concave and upper semicontinuous on M, there exists a (unique) maximizing element $v_i^* \in M$ which satisfies the Euler-Lagrange equations

$$\frac{\partial^2 v_i^*}{\partial x_3^2} = \frac{d}{dx_3}\left[L_{q+1}\left(\frac{2x_3}{d}\right)\right] \quad \text{in} \quad \left(-\frac{d}{2}, \frac{d}{2}\right),$$

$$\left.\frac{\partial v_i^*}{\partial x_3}\right|_{\pm d/2} = \begin{cases} +1 & \text{if } i = 1, \\ \pm 1 & \text{if } i = 2. \end{cases}$$

Hence we find that v_i^* is independent of (x_1, x_2) and given by

$$v_i^* = \frac{d}{2} \frac{L_{q+2}\left(\frac{2x_3}{d}\right) - L_q\left(\frac{2x_3}{d}\right)}{2q+3}, \quad q \ge 0, \tag{4.22}$$

and

$$\left(\Phi_i[v_i^*]\right)^2 = \frac{d}{2q+3} =: d\, C_{iq}. \tag{4.23}$$

Referring to (4.20), we have proved

<u>Theorem 4.1</u>. Assume that f_i in (4.3) is square integrable over ω. Then the error $\|e_i(q)\|_{E(\Omega)}$ for the hierarchical model of uniform order q (i.e. $\mathcal{P} = \{\omega\}$ and odd $q \ge 1$ for every $i = 1$, q even fo~ $i = 2$) can be estimated by

$$\gamma_0^2 \|e_i(q)\|_\varphi^2 \le \frac{d}{2q+3} \int_\omega \varphi^2 r_i^2 \, dx_1 dx_2 \tag{4.24}$$

where φ is as in Theorem 3.1 and

$$r_i(x_1,x_2) = f_i(x_1,x_2) - \frac{\partial u_i(q)}{\partial x_3}(x_1,x_2,x_3), \quad i = 1,2.$$

Based on (4.24) we define the indicator functions

$$\eta_{iq}(x_1,x_2) = \sqrt{\frac{d}{2q+3}}\,\varphi(x_1,x_2)\,r_i(x_1,x_2), \quad i = 1,2 \tag{4.25}$$

and the estimates $\mathcal{E}(u_i(q))$ defined in (4.8) are, according to (4.24), guaranteed upper estimates for $\|e_i(q)\|_{E(\Omega)}$, since $\varphi \equiv 1$ implies $\gamma_0 = 1$ in (3.9).

<u>Remark 4.2</u>. We emphasize that η_{iq} is very easy to compute, especially for low order models. We find in particular for $i = 2$, $q = 0$ that

$$\eta_{20}^2(x_1,x_2) = \frac{d}{3}\left(\varphi(x_1,x_2)f_2(x_1,x_2)\right)^2 .$$

Selecting $\varphi \equiv 1$ implies $Q = 0$ in Theorem 3.1, whence we obtain $\gamma_0 = 1$ in (3.9). Thus (4.24) yields with $r_2 = f_2 = (f^+ + f^-)/2$ the estimate

$$\|e_2(0)\|_{E(\Omega)}^2 \le \frac{d}{12}\|f^+ + f^-\|_{L^2(\omega)}^2 . \tag{4.26}$$ □

In the subsequent sections we will demonstrate that the estimators $\mathcal{E}(u_i(q))$ based on (4.25) are asymptotically and spectrally exact.

5. Asymptotic exactness of the error estimator.

Before demonstrating the asymptotic exactness of $\mathcal{E}$, we introduce some notation. Throughout, φ will denote the exponential weight function (4.11). Further

$$\|r\|^2_{k,\varphi} := \int_\omega |\nabla^k_x r|^2 \varphi^2 \, dx_1 dx_2 . \tag{5.1}$$

Finally, we introduce the class of data

$$T_\beta := \{f \,|\, \text{either } r_1(f) = 0 \text{ or } \|r_1\|_{1,\varphi}/\|r_1\|_{0,\varphi} \le \beta < \infty\}. \tag{5.2}$$

The main result on asymptotic exactness is

Theorem 5.1. Let θ_i, $i = 1,2$ denote the effectivity indices (4.9) with respect to the weighted energy norm (3.6). Assume further that $\mathcal{P} = \{\omega\}$, i.e. the model order is uniform. Then for $i = 1,2$ holds:

1.° If $f_i \in L^2(\omega)$ we have with $\Lambda_1 = \frac{2}{\pi}$, $\Lambda_2 = \frac{1}{\pi}$, that

$$\theta_i \ge \kappa_{i1} := \left[1-\Lambda_i \frac{d}{\sqrt{2}} Q\right]\left[1 + d\sqrt{2}\,\Lambda_i Q(1 + d\,\sqrt{2}\,\Lambda_i Q)\right]^{-1/2}, \quad i = 1,2 . \tag{5.3}$$

2.° If $f_i \in T_\beta$ then

$$\theta_i \le \kappa_{i2} := \left[1 + \frac{3}{2} d^2 D_q \,(\beta^2 + Q^2)\right]^{1/2}, \quad i = 1,2 \tag{5.4}$$

where D_q is given by

$$D_q = \frac{1}{(2q+3)^2 - 4} . \tag{5.5}$$

Moreover, if $\varphi \equiv 1$, the factor $3/2$ in κ_{12} can be replaced by $1/2$ and $Q = 0$.

Proof: 1° The bound (5.3) follows immediately from Theorems 4.1 and 3.1, if we note that $\Lambda_1 = 2/\pi$ and $\Lambda_2 = 1/\pi$ in Lemma 3.1 with $a = b = 1$ since the infimum there is taken only over odd, resp. even $\psi \in H^1(-1,1)$.

2° To show (5.4), we select in (4.12)

$$v = \bar{v}\varphi^2 = v_1^*(x_3)\, r_1(x_1,x_2)\varphi^2$$

with v_1^* as in (4.22) and get with (4.23) that

$$R_1(v) = dC_q \int_\omega r_1^2\varphi^2\, dx_1 dx_2 = B(e_1(q),v) \le \|e_1\|_\varphi\, \|v\|_{1/\varphi}\, . \tag{5.6}$$

Since

$$|\nabla_x v|^2 \le \varphi^2\Big\{3\varphi^2|\nabla_x\bar{v}|^2 + 6|\bar{v}|\ |\nabla_x\varphi|^2\Big\}$$

we find

$$\|v\|^2_{1/\varphi} = \int_\Omega \varphi^{-2}\Big\{|\nabla_x v|^2 + \Big(\frac{\partial v}{\partial x_3}\Big)^2\Big\}\, dx_1dx_2dx_3$$

$$\le \int_\Omega \Big\{3\varphi^2|\nabla_x\bar{v}|^2 + 6|\bar{v}|^2\ |\nabla_x\varphi|^2 + \varphi^2\ \Big(\frac{\partial\bar{v}}{\partial x_3}\Big)^2\Big\}\, dx_1dx_2dx_3$$

$$= \int_\Omega |v_1^*|^2\Big[3\varphi^2\ |\nabla_x r_1|^2 + 6|\nabla_x\varphi|^2\ |r_1|^2\Big] + \Big(\frac{dv_1^*}{dx_3}\Big)^2\ \varphi^2|r_1|^2\ \Big\}\, dx_1dx_2dx_3.$$

Since

$$\int_{-d/2}^{d/2} \Big(\frac{dv_1^*}{dx_3}\Big)^2 dx_3 = dC_q$$

and

$$\int_{-d/2}^{d/2} (v_1^*)^2\, dx_3 = \frac{1}{2}\, d^3C_qD_q\ ,$$

where D_q is as in (5.5), we get with $|\nabla_x\varphi|^2 \le Q^2\varphi^2/2$

$$(5.7) \qquad \|v\|^2_{1/\varphi} \le dC_q \, \|r_1\|^2_{0,\varphi} + \frac{3d^3}{2} C_q D_q \left\{ \|r_1\|^2_{1,\varphi} + Q^2 \, \|r_1\|^2_{0,\varphi} \right\} .$$

For every $\varepsilon > 0$ we have from (5.6)

$$2d \, C_q \|r_1\|^2_{0,\varphi} \le \varepsilon \, \|e_1\|^2_{\varphi} + \varepsilon^{-1} \|v\|^2_{1/\varphi} .$$

If we select $\varepsilon_0 > 0$ so that

$$(5.8) \qquad \varepsilon_0^{-1} \, \|v\|^2_{1/\varphi} \le d \, C_q \, \|r_1\|^2_{0,\varphi} = (\mathcal{E}(u_1))^2$$

we arrive at the desired (lower) bound

$$(\mathcal{E}(u_1(q))^2 \le \varepsilon_0 \, \|e_1(q)\|^2_{\varphi} \quad , \quad 1 = 1,2.$$

We estimate ε_0. Using (5.7) and (5.8) gives

$$\varepsilon_0 = \frac{\|v\|^2_{1/\varphi}}{dC_q \|r_1\|^2_{0,\varphi}} \le 1 + \frac{3}{2} d^2 D_q \left\{ Q^2 + \frac{\|r_1\|^2_{1,\varphi}}{\|r_1\|^2_{0,\varphi}} \right\}.$$

Using that $f_1 \in T_\beta$ gives (5.4). □

<u>Remark 5.1</u>. For $f_1 \in T_\beta$ and $\beta = \bar{\beta} d^{-\rho}$, $\rho < 1$ and the weight function φ defined in (4.11) with $\rho < 1$ we have from (5.3) and (5.4) that

$$\Theta \longrightarrow 1 \quad \text{for} \quad d \longrightarrow 0^+ .$$

Thus $\mathcal{E}_1$ based on (4.25) is a locally and asymptotically exact estimator for the modelling error. □

The local asymptotic exactness not only ensures that the indicator functions η_{1q} in (4.25) give a good estimate of the global modelling error in energy norm, but also for subdomains $\tilde{\omega}$ of ω that

$$\mathscr{E}_{1q}(\rho;\tilde{\omega}) := \sqrt{\frac{d}{2q+3}} \left[\int_{\mathrm{dist}(x_1,x_2;\tilde{\omega})<d^{\rho}} r_1^2(x_1,x_2)\, dx_1 dx_2 \right]^{\frac{1}{2}}$$

is an asymptotically exact measure for the local contribution to the modelling error at $\tilde{\omega}$.

It further guarantees that a local increase of the model order in $\tilde{\omega}$ will reduce the error in $\tilde{\omega}$ while leaving the error elsewhere in $\omega\backslash\tilde{\omega}$ unchanged -- a feature typically not found for elliptic equations and a consequence of the fact that Ω is a thin domain. These observations are the basis for the adaptive selection of the model orders on subdomains of ω [8].

6. Spectral exactness of the error estimator.

Our purpose in the present section is to show the spectral exactness of the errcr estimator $\mathscr{E}$, i.e. that κ_{11}, κ_{12} in Theorem 5.1 tend to one as $q \longrightarrow \infty$ at fixed $d > 0$. While this is not hard to establish for κ_{12} in (5.4), the corresponding proof for κ_{11} requires a more careful analysis of the constant Λ in Lemma 3.1.

We denote by $\varphi_k(x_1,x_2)$ the eigenfunctions (orthonormalized in $L^2(\omega)$) of the eigenvalue problem

$$(6.1) \qquad -\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in} \quad \omega\ , \quad \varphi_k = 0 \quad \text{on} \quad \partial\omega\ ,$$

$k = 1,2,3...$ and the eigenvalues are enumerated with respect to increasing magnitude and counting multiplicity. We collect some of their properties that will be needed later.

Lemma 6.1. Assume that $\omega \subset \mathbb{R}^n$, $n = 1,2$ and that $\partial\omega$ is smooth if $n = 2$. Then 1.° $\lambda_1 > 0$ is a simple eigenvalue and the corresponding eigenfunction

$\varphi_1(x_1,x_2) > 0$ in ω.

2° For all $k \in \mathbb{N}$, and all $(x_1,x_2) \in \bar{\omega}$,

$$\left\|\frac{\varphi_k}{\varphi_1}\right\|_{C^0(\bar{\omega})} \le C(\omega)\theta_k \ , \ \theta_k = \begin{cases}\lambda_k^{1/2} & \text{if} \quad \omega \subset \mathbb{R}^1 \\ \lambda_k^{1+\delta} & \text{if} \quad \omega \subset \mathbb{R}^2\end{cases} . \tag{6.2}$$

Proof. Assertion 1° is well known. Let us prove 2°. In the case $n = 1$ the assertion follows from the explicitly known eigenfunctions. We consider therefore $n = 2$ and claim that

$$-\left.\frac{\partial\varphi_1}{\partial n}\right|_\gamma \ge C_1(\omega) > 0 .$$

To prove it, we note that $\varphi_1 > 0$ in ω and hence the function $u = -\varphi_1$ satisfies

$$\Delta u = \lambda_1\varphi_1 \ge 0 \text{ in } \bar{\omega} \ , \ u < 0 \text{ in } \omega, \ u = 0 \text{ on } \partial\omega.$$

Now we can apply the maximum principle [12, Theorem 2.7] and find that $\frac{\partial u}{\partial n}(x_0) > 0$ for all $x_0 \in \partial\omega$ from where the claim follows.

Now, since γ is smooth, there exists $\varepsilon_0(\omega) > 0$ such that $\omega_\varepsilon := \{x \in \omega \,|\, \mathrm{dist}(x,\gamma) > \varepsilon\}$ is smooth, too, for $0 < \varepsilon < \varepsilon_0$. Hence we can estimate, possibly after reducing ε_0,

$$\varphi_1(x) \ge \frac{1}{2} C_1(\omega)\varepsilon \quad \text{on} \quad \omega_\varepsilon$$

and get

$$\begin{aligned}\left\|\frac{\varphi_k}{\varphi_1}\right\|_{L^\infty(\omega_\varepsilon)} &\le \frac{2}{\varepsilon_0 C_1(\omega)} \|\varphi_k\|_{L^\infty(\omega_\varepsilon)} \\ &\le \frac{2}{\varepsilon_0 C_1(\omega)} C_2(\omega) \|\nabla\varphi_k\|_{L^2(\omega)} ,\end{aligned}$$

i.e.

$$\left\|\frac{\varphi_k}{\varphi_1}\right\|_{L^\infty(\omega_\varepsilon)} \le C_3(\omega)\sqrt{\lambda_k}\,\|\varphi_k\|_{L^2(\omega)} = C_3\sqrt{\lambda_k} . \tag{6.3}$$

In $\sigma_\varepsilon := \bar{\omega}\backslash\omega_\varepsilon$ we introduce the boundary-fitted coordinates (ρ, s). Then we have from Taylor's formula that

$$\frac{|\varphi_k(\rho,s)|}{|\varphi_1(\rho,s)|} = \frac{|\rho\frac{\partial\varphi_k}{\partial\rho}(0,s) + O(\rho^2)|}{|\rho\frac{\partial\varphi_1}{\partial\rho}(0,s) + O(\rho^2)|} \le C_4(\omega) \sup_{x\in\gamma} \left\{ \frac{\left|\frac{\partial\varphi_k}{\partial n}(x)\right|}{\left|\frac{\partial\varphi_1}{\partial n}(x)\right|} \right\}$$

$$\le \frac{C_4(\omega)}{C_1(\omega)} \left\| \frac{\partial\varphi_k}{\partial n} \right\|_{L^\infty(\gamma)} \le \frac{C_4}{C_1} C_5(\omega) \left\| \frac{\partial\varphi_k}{\partial n} \right\|_{H^{1/2+\delta}(\gamma)}$$

$$\le C_6(\omega) \|\varphi_k\|_{H^{2+2\delta}(\omega)}$$

where C_6 is independent of k, but depends on $\delta > 0$. Hence

$$\left\| \frac{\varphi_k}{\varphi_1} \right\|_{C_0(\bar{\sigma}_\varepsilon)} \le C_7(\omega) \left\| \Delta^{1+\delta}\varphi_k \right\|_{L^2(\omega)} = C_7(\omega)\, \lambda_k^{1+\delta} . \tag{6.4}$$

Combining (6.3) and (6.4) completes the proof. □

With the eigenfunctions

$$\psi_{1m}\left(\frac{2y}{d}\right) = \sin\left(\mu_{1m}\frac{2y}{d}\right), \quad \mu_{1m} = \frac{2m-1}{2}\pi \tag{6.5}$$

and

$$\psi_{2m}\left(\frac{2y}{d}\right) = \cos\left(\mu_{2m}\frac{2y}{d}\right), \quad \mu_{2m} = m\pi \tag{6.6}$$

$m = 1,2,3,\dots$, the eigenfunctions for $-\Delta$ on Ω with homogeneous boundary conditions on Γ and $R_\pm$ which satisfy (2.13) are given by

$$\Phi_{ikm} := \varphi_k(x_1,x_2)\psi_{im}\left(\frac{2x_3}{d}\right), \tag{6.7}$$

with the corresponding eigenvalues

$$\Lambda_{ikm} = \lambda_k + 4d^{-2}\mu_{im}^2, \quad i = 1,2 \quad k,\ m \in \mathbb{N}. \tag{6.8}$$

The sequence of eigenfunctions Φ_{ikm} is dense in the space H defined in (3.5). Therefore we have in particular for the modelling errors e_i the expansion

$$e_i = \sum_{k,m\in\mathbb{N}} E_{ikm}\,\Phi_{ikm}\,, \quad i = 1,2 \tag{6.9}$$

where the coefficients can be determined from Lemma 4.1.

<u>Lemma 6.2</u>.

$$E_{ikm} = \left(\frac{d}{2}\lambda_k + \frac{2}{d}\mu_{im}^2\right)^{-1}\beta_{im}^q\,\rho_{ik}$$

with ρ_{ik} and β_{im}^q as in (6.10) and (6.11) below, respectively.

Proof. We recall (4.12)

$$B(e_i,v) = R_i(v) \qquad \forall v\in H \quad i = 1,2,$$

where, with *(4.16) and integration by parts,*

$$R_i(v) = \int_\omega r_i(x_1,x_2)\int_{-d/2}^{d/2}\frac{\partial v}{\partial x_3}\,L_{q+1}\left(\frac{2x_3}{d}\right)\,dx_3dx_1dx_2\;.$$

Since

$$B\left[\Phi_{ikm},\,\phi_{jln}\right] = \left[\frac{d}{2}\lambda_k + \frac{2}{d}\mu_{im}^2\right]\delta_{ij}\delta_{kl}\delta_{mn}$$

we obtain with

$$\rho_{ik} = \int_\omega r_i(x_1,x_2)\;\varphi_k(x_1,x_2)\;dx_1dx_2 \tag{6.10}$$

and

$$\beta_{im}^q = \int_{-1}^{1}\psi_{im}'(z)\;L_{q+1}(z)\;dz \tag{6.11}$$

the assertion. □

Our purpose is to estimate the dependence of Λ in Lemma 3.1 on the model order q since

(6.12)
$$\inf_{(x_1,x_2)\in\omega} \inf_{e_1} \frac{\int_{-d/2}^{d/2} \left(\frac{\partial e_1}{\partial_3}\right) dx_3}{\int_{-d/2}^{d/2} (e_1)^2 \, dx_3} \ge \frac{1}{(\Lambda_1(q)d)^2}$$

where the infimum is taken over all e_1 of the form (6.9). We compute

$$\int_{-d/2}^{d/2} (e_1)^2 \, dx_3 =$$

(6.13)
$$\left(\frac{d}{2}\right)^3 \sum_{k,\ell,m} \rho_{1k}\rho_{1\ell} \left[1 + \frac{d^2}{4}\frac{\lambda_k}{\mu_{1m}^2}\right]^{-1} \left[1 + \frac{d^2}{4}\frac{\lambda_1}{\mu_{1m}^2}\right]^{-1} \frac{(\beta_{1m}^q)^2}{(\mu_{1m})^4} \varphi_k\varphi_\ell$$

and

$$\int_{-d/2}^{d/2} \left(\frac{\partial e_1}{\partial x_3}\right)^2 dx_3 =$$

(6.14)
$$\left(\frac{d}{2}\right) \sum_{k,\ell,m} \rho_{1k}\rho_{1\ell} \left[1 + \frac{d^2}{4}\frac{\lambda_k}{\mu_{1m}^2}\right]^{-1} \left[1 + \frac{d^2}{4}\frac{\lambda_1}{\mu_{1m}^2}\right]^{-1} \frac{(\beta_{1m}^q)^2}{(\mu_{1m})^2} \varphi_k\varphi_\ell \, .$$

To obtain a lower bound for (6.12), we estimate (6.13) from above and (6.14) from below (pointwise). For (6.14) we have with $a = 1 + d^2\lambda_1/(1\mu_{11}^2)$

$$\int_{-d/2}^{d/2} \left(\frac{\partial e_1}{\partial x_3}\right)^2 dx_3 \ge$$

$$\frac{d}{2}\left\{|\rho_{11}|^2|\varphi_1|^2 a^{-2} - 2|\varphi_1|\ |\rho_{11}| \sum_{k\ge 2} \left[1 + \frac{d^2}{4}\frac{\lambda_k}{\mu_{11}^2}\right]^{-1} |\rho_{1k}|\ |\varphi_k|\right.$$

$$\left. - \sum_{k,\ell\ge 2} \left[1 + \frac{d^2}{4}\frac{\lambda_k}{\mu_{11}^2}\right]^{-1} \left[1 + \frac{d^2}{4}\frac{\lambda_\ell}{\mu_{1m}^2}\right]^{-1} |\rho_{1k}|\ |\rho_{1\ell}|\ |\varphi_k|\ |\varphi_\ell|\right\} \frac{(\beta_{1m}^q)^2}{(\mu_{1m})^2} =$$

$$= \frac{d}{2} a^{-2} |\rho_{11}|^2 |\varphi_1|^2 \left\{1 - 2\sum_{k\ge 2} \left[1 + \frac{d^2\lambda_k}{4\mu_{11}^2}\right]^{-1} \frac{|\rho_{1k}|}{|\rho_{11}|} \frac{|\varphi_k|}{|\varphi_1|}\right.$$

$$- \left[\sum_{k\geq 2} \left(1 + \frac{d^2\lambda_k}{4\mu_{11}^2}\right)^{-1} \frac{|\rho_{1k}|}{|\rho_{11}|} \frac{|\varphi_k|}{|\varphi_1|} \right]^2 \Bigg\} \sum_m \frac{(\beta_{im}^q)^2}{(\mu_{im})^2}$$

$$\text{(6.15)} \qquad = \frac{d}{2}\, a^{-2} |\rho_{11}|^2\, |\varphi_1|^2\, (1 - 2\psi - \psi^2) \sum_m \frac{(\beta_{im}^q)^2}{(\mu_{im})^2}$$

where we assumed that $\rho_{11} \neq 0$ and defined

$$\text{(6.16)} \qquad \Psi := C(\omega) \sum_{k\geq 2} \left[1 + \frac{d^2\lambda_k}{4\mu_{11}^2}\right]^{-1} \theta_k \frac{|\rho_{1k}|}{|\rho_{11}|}$$

with $C(\omega)$ and θ_k as in (6.2).

Consider next the upper bound for (6.13). We estimate analogously as before and get

$$\text{(6.17)} \qquad \int_{-d/2}^{d/2} (e_i)^2\, dx_3 \leq \left(\frac{d}{2}\right)^3 |\varphi_1|^2 |\rho_{11}|^2\, a^{-2}\, (1 + \Psi)^2 \sum_m \frac{(\beta_{im}^q)^2}{(\mu_{im})^4}$$

$i = 1,\ q = 2m + 1$ or $i = 2,\ q = 2m$.

Now

$$\beta_{im}^q = \int_{-1}^{1} \psi_{im}'\, L_{q+1}(z)\, dz$$

$$= - \int_{-1}^{1} \psi_{im}'' \int^z L_{q+1}(\xi) d\xi dz + \psi_{im}' \int^z L_{q+1}\, d\xi \Bigg|_{z=\pm 1}$$

$$= \mu_{im}^2 \int_{-1}^{1} \psi_{im} \int^z L_{q+1}(\xi) d\xi\, dz$$

where we used that $\psi_{im}'(\pm 1) = 0$ and $-\psi_{im}'' = \mu_{im}^2\, \psi_{im}$. Hence $(\beta_{im}^q)^2 / \mu_{im}^2$ are the Fourier coefficients of the antiderivative of L_{q+1}, i.e. of

$$(2q+3)^{-1}(L_{q+2}(z) - L_q(z))\;.$$

Therefore

$$\sum_m \frac{(\beta^q_{1m})^2}{(\mu_{1m})^4} = K(2q+3)^{-2}\,\|L_{q+2} - L_q\|^2_{L^2(-1,1)} \tag{6.18}$$

$$= K\,\frac{4}{((2q+3)^2-4)(2q+3)}$$

and analogously

$$\sum_m \frac{(\beta^q_{1m})^2}{(\mu_{1m})^2} = K\|L_{q+1}\|^2_{L^2(-1,1)} = K\,\frac{2}{2q+3} \tag{6.19}$$

where K is a constant depending on the normalization of the ψ_{1m} in (6.5), (6.6) (its numerical value is immaterial in what follows). We can now prove

Theorem 6.1.

1° If $f_1 \in T_\beta$ defined in (5.2) with $\beta = \bar\beta(q/d)^{1-\varepsilon}$, $\varepsilon > 0$, and $\bar\beta$ independent of d and q, then

$$\kappa_{12} \to 1^+ \quad \text{both as} \quad d \to 0^+\,,\; q \to \infty\;.$$

2° If, moreover, f_1 is such that

α) $\rho_{11} \neq 0$

β) $\Psi(r_1, \omega, d, q) \le (\sqrt{2} - 1)(1 - D_q^{1-\varepsilon})$ for some $\varepsilon > 0$, then

$$\kappa_{11} \to 1 \quad \text{both as} \quad d \to 0^+\,,\; q \to \infty.$$

Proof: Assertion 1° follows immediately with the definition (5.4) of κ_{12} and the assumption on β. To show 2°, we note that κ_{11} is as in (5.3), however now with Λ_1 determined from (6.12) instead of Lemma 3.1. Using (6.15), (6.17) in (6.12) we find

$$\frac{1}{(\Lambda_1(q)d)^2} \ge \frac{4}{d^2}\,\frac{1-2\Psi-\Psi^2}{1+2\Psi+\Psi^2}\,\frac{\sum_m (\beta^q_{1m})^2(\mu_{1m})^{-2}}{\sum_m (\beta^q_{1m})^2(\mu_{1m})^{-4}}$$

and, from (6.18), (6.19) with $\Psi = (\sqrt{2}-1) - \delta$, $0 < \delta < \sqrt{2} - 1$,

$$\frac{1}{(\Lambda_1(q)d)^2} \ge \frac{4}{d^2}\,\frac{\delta(2\sqrt{2}-\delta)}{(\sqrt{2}-\delta)^2}\,\frac{(2q+3)^2-4}{2}$$

$$\ge \frac{\delta}{d^2 D_q}(1+\sqrt{2}) \ .$$

Hence

$$\Lambda_1(q) \le (1+\sqrt{2})^{-1} D_q/\delta$$

and, using $\delta = (\sqrt{2} - 1)\, D_q^{1-\varepsilon}$, we find in (5.3)

$$\Lambda_1(q) \le D_q^{\varepsilon} = ((2q+3)^2 - 4)^{-\varepsilon} \ .$$

This completes the proof. $\square$

Therefore, under the assumptions made, Theorems 5.1 and 6.1 establish the asymptotic and spectral exactness of the estimator $\mathcal{E}$ in (4.8).

7. A posteriori error estimation in the L^2-norm

In the present section, we derive a-posteriori estimators for $\|e_1\|_{L^2(\Omega)}$. To this end we consider the bilinear form

$$B_1(u,v) = \int_\Omega u\, \Delta v\, dx \tag{7.1}$$

on $H_1^* \times H_2^*$ where

$$H_1^* = \left\{ u \in L^2(\Omega) \ \middle|\ \int_{-d/2}^{d/2} u(x_1,x_2,x_3)\, dx_3 = 0 \ \text{ a.e. } (x_1,x_2) \in \omega \right\},$$

furnished with the norm

$$\|u\|_1 = \left(\int_\Omega |u|^2\, dx\right)^{\frac{1}{2}} = \|u\|_{L^2(\Omega)}$$

and where

$$H_2^* = \left\{ v \in H_\varphi(\Omega) \;\middle|\; \|\Delta v\|_{L^2(\Omega)} < \infty\ ,\ \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad R_\pm \right\}$$

with H_φ defined in (3.3), furnished with the norm

$$\|v\|_2 = \left(\int_\Omega |\Delta v|^2\, dx\right)^{1/2} = \|\Delta v\|_{L^2(\Omega)}.$$

We note that locally $v \in H^2(\Omega)$ and hence $\frac{\partial v}{\partial n}$ is well defined. Furthermore, it is also readily seen that $\|\circ\|_2$ is a norm on H_2^*.

<u>Theorem 7.1</u>. The bilinear form $B_1(u,v)$ in (7.1) satisfies (3.1) and (3.2) with $C = \gamma = 1$.

Proof. It is easy to see that (3.1) holds with $C = 1$. We will now estimate γ in (3.2). For given $u \in H_1^*$, define s to be the solution of

(7.2) $$\Delta s = u \qquad \text{in} \quad \Omega,$$

(7.3) $$s = 0 \qquad \text{on} \quad \Gamma,$$

(7.4) $$\frac{\partial s}{\partial n} = 0 \qquad \text{on} \quad R_\pm\ .$$

Since $u \in L^2(\Omega)$, s obviously exists and is uniquely determined. Define

(7.5) $$z := s - \frac{1}{d}\int_{-d/2}^{d/2} s(x_1,x_2,x_3)\, dx_3,$$

Therefore

$$B_1(u,v) = \int_\Omega u\Delta z\, dx = \|u\|_1^2,\ \|z\|_2 = \|u\|_1$$

and (3.2) follows with $\gamma = 1$. □

We turn now to the derivation of the a-posteriori estimator for $\|e(q)\|_1 = \|e(q)\|_{L^2(\Omega)}$. We start from the characterization

$$B_1(e_1(q),v) = R_1(v) \tag{7.6}$$

with $R_1(\cdot)$ as in (4.16). Using Theorem 7.1 gives

$$\|e_1\|_{L^2(\Omega)} = \sup_v \frac{|R_1(v_1)|}{\|\Delta v\|_{L^2(\Omega)}}$$

where the supremum is taken over all $0 \neq v \in H_2^* \cap H_1$ which satisfy (2.13) with $b = 1$. To estimate the supremum, we observe that any $v \in H_2^*$ can be written in the form

$$v_i(x_1,x_2,x_3) = \sum_{k,m\geq 1} a_{ikm}\varphi_k(x_1,x_2)\psi_{im}\left(\frac{2x_3}{d}\right) \tag{7.7}$$

where φ_k and ψ_{im} are as in (6.5)-(6.8). Further, we find that

$$-\Delta v_i = \sum_{k,m\geq 1} b_{ikm}\ \varphi_k(x_1,x_2)\psi_{im}\left(\frac{2x_3}{d}\right) \tag{7.8}$$

where

$$b_{ikm} = a_{ikm}\ \Lambda_{ikm} \qquad i = 1,2,\ k,m \in \mathbb{N},$$

with Λ_{ikm} as in (6.8). We insert (7.7) in $R_i(v_i)$ in (4.12) and find for $i = 2$ with $q = 2m$ that

$$R_2(v_2) = \int_\Omega r_2(x_1,x_2)\Big\{v_2(x_1,x_2,d/2) + v_2(x_1,x_2,-d/2) +$$

$$\sum_{j=0}^{2m} \Lambda_{2,2j} \int_{-d/2}^{d/2} v_2(x_1,x_2,x_3)L_{2j}\left(\frac{2x_3}{d}\right)dx_3\Big\}\ dx_1dx_2$$

where

$$\Lambda_{2,2j} = -\frac{2}{d}\,(4j+1)\ .$$

Next, using (7.7) yields

$$R_2(v_2) = 2 \sum_{k,\ell} a_{2k\ell}\, \rho_{2k}\, ((-1)^{\ell} - \tau_{2q\ell})$$

where

$$\tau_{20\ell} = 0 \qquad \forall \ell \qquad ,$$

(7.9)

$$\tau_{2q\ell} := \frac{1}{2} \sum_{j=0}^{m} (4j+1) \int_{-1}^{1} L_{2j}(z) \cos(\ell\pi z)\, dz$$

where $q = 2m > 0$. Therefore

$$|R_2(v_2)|^2 \le 4 \sum_k (\rho_{2k}) \sum_k C_k^2$$

where

$$C_k = \sum_{\ell} b_{2k\ell}(\Lambda_{2k\ell})^{-1}((-1)^{\ell} - \tau_{2q\ell})$$

and we estimate

$$C_k^2 \le \sum_{\ell} (b_{2k\ell})^2 \sum_{\ell} (1 - \tau_{2q\ell})^2 (\Lambda_{2k\ell})^{-2} .$$

Since from (7.7)

(7.10)
$$\|\Delta v_1\|^2_{L^2(\Omega)} = \frac{d}{2} \sum_{k,m} (b_{1km})^2$$

we find

$$\|e_2\|^2_{L^2(\Omega)} = \sup_{v_2} \frac{|R_2(v_2)|^2}{\|\Delta v_2\|^2_{L^2(\Omega)}}$$

$$\le \frac{1}{2} d^3 \|r_2\|^2_{L^2(\omega)} \sum_{\ell} ((-1)^{\ell} - \tau_{2q\ell})^2 (\mu_{2\ell})^{-4} .$$

With an analogous reasoning for $i = 1$, we have shown

Theorem 7.2. For $i = 1,2$

$$\|e_i\|_{L^2(\Omega)} \le \sqrt{E_{iq}}\, d^{3/2} \|r_i\|_{L^2(\omega)}$$

where

$$E_{iq} = \frac{1}{2} \sum_{\ell} ((-1)^{\ell} - \tau_{iq\ell})^2 (\mu_{i\ell})^{-4}$$

and $\tau_{2q\ell}$ is as in (7.9), and for $i = 1$, $q = 2m+1$,

$$\tau_{1q\ell} = \frac{1}{2} \sum_{j=0}^{m} (4j+3) \int_{-1}^{1} L_{2j+1}(z) \sin(\mu_{1\ell} z)\, dz.$$

Remark 7.1.

For $q = 0$, $\tau_{20\ell} = 0$ and hence $E_{20} = \frac{1}{2} \pi^{-4} \zeta(4) = 1/180$. For $q > 0$, E_{iq} must be computed numerically in general.

8. Error estimation for laminated materials.

The analysis in the preceding sections carries over to the general problem (1.1) with minor modifications which we will describe. To underline the analogy, we assume that a basis ψ_j has been selected so that its span coincides with that obtained from (2.6)-(2.9), and further, that for $j = 0,1,2,\ldots$

(8.1) $$\psi_{2j}(z) = \psi_{2j}(-z),\ \psi_{2j+1}(z) = -\psi_{2j+1}(-z) ,$$

with

(8.2) $$\int_{-1}^{1} b(z)\, \psi_j(z)\psi_k(z)\, dz = \delta_{jk} \frac{2}{2j+1} ,$$

the normalization satisfied by $L_j(z)$. Then we obtain, because of the way span $\{\psi_j\}$ was defined, that for $i = 1,2$

$$R_i(v) = \int_\omega r_i(x_1,x_2)\left\{v(x_1,x_2,d/2) \pm v(x_1,x_2,-d/2)\right\} dx_1dx_2$$

(8.3)

$$+\sum_{j=0}^{q} a_{ij}\int_\omega r_i(x_1,x_2)\int_{-d/2}^{d/2} b\left(\frac{2x_3}{d}\right)\psi_j\left(\frac{2x_2}{d}\right)v\ (x_1,x_2,x_3)\ dx_3dx_1dx_2,$$

where "-" corresponds to $i = 1$, "+" to $i = 2$. Since $R_i(v) = 0$ for all $v \in S(q)$, we find from (8.2) that

$$a_{ij} = -\frac{2}{d}\psi_j(1)\ (2j+1) \qquad \begin{array}{l} 0 \le j \le q \\ i = 1 \text{ if } j \text{ is odd,} \\ i = 2 \text{ if } j \text{ is even.}\end{array}$$

The residuals r_i are defined for $i = 1,2$ by

(8.4) $$r_i(x_1,x_2) := f_i(x_1,x_2) - a(1)\frac{\partial u_i(\mathcal{P},q)}{\partial n}(x_1,x_2,d/2)\ .$$

Assuming $\mathcal{P} = \{\omega\}$, we reason as in (4.18)-(4.20) and find

(8.5) $$\gamma_0\ \|e_i(q)\|_\varphi^2 \le \int_\omega \varphi^2 r_i^2 \sup_{v\in M}(\Phi_i[v])^2\ dx_1dx_2$$

where

$$\Phi_i[v]\ (x_1,x_2) :=$$

$$\frac{v(x_1,x_2,d/2) \pm v(x_1,x_2,-d/2) - \sum_{j=0}^{q} a_{ij}\int_{-d/2}^{d/2} b\left(\frac{2x_3}{d}\right)\psi_j\left(\frac{2x_3}{d}\right)v\ dx_3}{\left[\int_{-d/2}^{d/2} a\left(\frac{2x_3}{d}\right)\left(\frac{\partial v}{\partial x_3}\right)^2(x_1,x_2,x_3)\ dx_3\right]^{1/2}}$$

where the supremum is taken over

$$M := L^2(\omega;H^1(-d/2,d/2)) \cap \left\{v \ \middle|\ \int_{-d/2}^{d/2} b\left(\frac{2x_3}{d}\right)v\ dx_3 = 0 \ \text{ a.e. } (x_1,x_2) \in \omega\right\}.$$

Once more the variational problems $\sup_M \Phi_i[v]$ admit unique maximizers v_i^*

which are independent of (x_1, x_2) and satisfy the Euler Lagrange equations (in the weak sense)

(8.6)

$$\frac{\partial}{\partial x_3}\left[a\left(\frac{2x_3}{d}\right)\frac{\partial v_i^*}{\partial x_3}\right] = \sum_{j=0}^{q} a_{ij} b\left(\frac{2x_3}{d}\right)\psi_j\left(\frac{2x_3}{d}\right),$$

$$a(\pm 1)\left.\frac{\partial v_i}{\partial x_3}\right|_{\pm d/2} = \begin{cases} +1 & x_3 = d/2 \qquad i = 1 \\ \pm 1 & x_3 = -d/2 \qquad i = 2 . \end{cases}$$

Defining

(8.7) $$(\Phi_i[v_i^*])^2 := d\, C_{iq} \qquad \begin{array}{ll} i = 1,\ q \ge 0, & \text{even} \\ i = 2,\ q \ge 1, & \text{odd} \end{array}$$

where C_{iq} is independent of d (and must generally be calculated numerically), we have proved

<u>Theorem 8.1</u>. Under the assumptions of Theorem 4.1 we have, for $i = 1,2$,

(8.8) $$\gamma_0 \|e_i\|(q)\|_\varphi^2 \le d\, C_{iq} \int_\omega \varphi^2 r_i^2\, dx_1 dx_2$$

where φ is as in Theorem 3.1 and r_i as in (6.4).

The indicator functions are therefore now

(8.9) $$\eta_{iq}(x_1,x_2) = \sqrt{d\, C_{iq}}\; \varphi(x_1,x_2)\; r_i(x_1,x_2) .$$

It can also be shown that exact analogs of Theorems 5.1 and 6.1 hold with suitably modified constants D_q. We shall, however, not elaborate since the details are completely analogous.

<u>Remark 8.1</u>. In the practically important case that $a(z)$, $b(z)$ are piecewise constant functions, C_{iq} in (8.7) can be computed numerically by maximizing $\Phi[v]$ over piecewise polynomials in one dimension (of sufficiently high degree).

9. A simple example of a-posteriori error estimation

We consider

$$\Omega = (-1,1) \times (-d/2,\ d/2) \tag{9.1}$$

and select $f^{+} = f^{-}$ in (1.1) so that

$$u = u(x,y) = 2 \cos \left(\frac{\pi}{2}\, x\right) \cosh \left(\frac{\pi}{2}\, y\right) .$$

Then, for uniform model order $q = 2m$

$$u(q) = \sum_{j=0}^{q} U_j(x) L_{2j} \left(\frac{2y}{d}\right) = \sum_{j=0}^{m} x_j \cos \left(\frac{\pi}{2}\, x\right) L_{2j} \left(\frac{2y}{d}\right).$$

Then the vector $\underline{x} = (x_0, \ldots\ldots, x_m)^T$ is determined from

$$\left(\frac{d\pi}{8}\, \underline{A} + \frac{2}{d}\, \underline{B}\right) \underline{x} = \alpha\, \underline{e}$$

where

$$\alpha = 2\pi \sinh \left(\frac{\pi d}{4}\right), \quad \underline{e} = (1, \ldots, 1)^T,$$

and

$$\underline{A}_{ij} = \int_{-1}^{1} L_{2i} L_{2j}\, dz, \quad \underline{B}_{ij} = \int_{-1}^{1} L'_{2i}\, L'_{2j}\, dz .$$

Selecting the weight function $\varphi = 1$, we find

$$\| e(q) \|^2_{1,\varphi} = \alpha \left[2 \cosh \left(\frac{\pi d}{4}\right) - \underline{x}^T \underline{e} \right]$$

and the estimator

$$\mathcal{E}^2 = \frac{d}{4(2q+3)} \left[\alpha - \frac{4}{d} \sum_{i=1}^{m} x_j L'_{2j}(1) \right]^2 .$$

Using a computer algebra system, we obtain

$$\theta^2 := \mathcal{E}^2 / \| e(q) \|^2_{1,\varphi} = 1 + d^2\, \frac{\pi^2}{m_q} + \theta(d^4)$$

where m_q is listed in Table 9.1.

q	0	2	4	6	8	10	12
m_q	240	360	936	1768	2856	4200	5800

Table 9.1. m_q is the asymptotic expansion of the effectivity index.

Not only is $\mathcal{E}$ asymptotically exact as predicted in Theorem 5.1, but we observe that with $Q = 0$ and $\beta = \pi/2$ we have $\kappa_{21} = 1$ in (5.3) and in (5.4)

$$(\kappa_{12})^2 = 1 + \frac{d^2\pi^2}{8\{(2q+3)^2 - 4\}}$$

and a comparison with Table 9.1 shows that for $q \geq 2$ this bound for κ_{22} is the best possible one.

Further, it is verified directly that $\varphi_1 = \cos\left(\frac{\pi}{2}x\right)$ in this case, hence Ψ in (6.16) is equal to zero, and by Theorem 6.1 we have in (5.3) that

$$(\Lambda_i)^2 \leq \frac{1}{2} D_q , \qquad i = 1,2,$$

i.e. $\mathcal{E}$ is spectrally exact for weight functions φ satisfying

$$Q = O(q^\rho d^{-\rho}) \quad , \quad 0 \leq \rho < 1.$$

Finally, in Table 9.2 we present the asymptotic expansion as $d \to 0$ of

$$d^3\|r\|^2_{L^2(\omega)} \Big/ \|e(q)\|^2_{L^2(\Omega)} .$$

q	$d^3 \|r\|^2_{L^2(\omega)} \Big/ \|e(q)\|^2_{L^2(\Omega)}$
0	$180 + 15(\pi d)^2 / 7 + O(d^4)$
2	$630 + 315(\pi d)^2 / 44 + O(d^4)$
4	$2574 + 1287(\pi d)^2 / 140 + O(d^4)$
6	$6630 + 9945(\pi d)^2 / 836 + O(d^4)$

Table 9.2. Asymptotics of the L^2-residual versus the L^2-error for small q.

In each case the leading term agrees with the numerical value for E_{2q} obtained from Theorem 7.1 which shows the asymptotic exactness of the estimator there for our model problem.

Further, in the unweighted case (i.e. $\varphi \equiv 1$) we find that $f \in T_\beta$ with $\beta = \sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of $-\frac{d^2}{dx^2}$ in $(-1,1)$ with boundary conditions $u(\pm 1) = 0$, so that we have here

$$\kappa_{11} = 1 \le \Theta \le \left[1 + \frac{\pi^2}{8} d^2 D_q\right]^{1/2} = \kappa_{12}$$

with D_q as in (5.5), i.e. for this problem the estimator (4.8) with (4.25) is asymptotically and spectrally exact.

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